Compactification of moduli spaces in quantum toric geometry

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Abstract

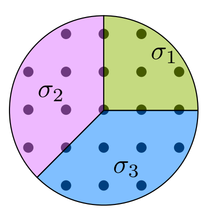
A toric variety is a complex variety which is completely described by the combinatorial data of a fan of strongly convex rational (with respect to a lattice) cones. Due to this rationality condition, toric varieties are (equivariantly) rigid since if we deform a lattice, it can become dense. A solution to this problem is to consider a stacky generalization of toric varieties where the "lattice" is, in fact, a finitely generated subgroup of $\mathbb{R}^d$ (in the simplicial case as introduced by L. Katzarkov, E. Lupercio, L. Meersseman and A. Verjovsky). The goal of this talk is to explain the moduli spaces of quantum toric stacks and their compactification.

Introduction

0.1 Classical toric varieties

Toric variety = normal complex variety with an action of an algebraic torus $(\mathbb{C}^*)^d$ having a dense orbit

Main interest : Fully described by fans i.e. families of rational strongly convex cones (stable by intersection and taking faces).



Each maximal cone represents a chart of $\mathbb{P}^2$ and the intersection of two such cones give the transition (monomial) map.

Theorem 0.1. *This correspondance is an equivalence of categories* **Fans** $\to$ **Torics**

0.2 Moduli spaces and rationality condition

Rationality condition $\Rightarrow$ toric varieties are rigid as equivariant spaces :

The continuous deformation of cones and their underlying lattice leads to dense subgroups of $\mathbb{R}^d$

Example 0.2.

$$\Gamma_\alpha = \mathbb{Z}^2 + \alpha\mathbb{Z} \begin{cases} \text{is discrete and of rank 2 if } \alpha \in \mathbb{Q}^2 \\ \text{is not discrete and can be dense in } \mathbb{R}^2 \text{ otherwise} \end{cases}$$

If $\alpha = (1, \pi)$, $\overline{\Gamma_\alpha} \simeq \mathbb{R} \times \mathbb{Z}$ and if $\alpha = (\sqrt{2}, \sqrt{3})$ then $\overline{\Gamma_\alpha} = \mathbb{R}^2$.

$\rightsquigarrow$ No moduli spaces of toric varieties.

We need to consider more general objects : "quantum toric stacks" as introduced by Ludmil Katzarkov, Ernesto Lupercio, Laurent Meersseman, Alberto Verjovsky.

Theorem 0.3 (KLMV,B)**.** *There exists a fine moduli space of quantum toric stacks with fixed combinatorics i.e. a moduli space with a universal family.*

(extending the equivalence of categories between fans and toric varieties).

Theorem 0.4 (B)**.** *These moduli spaces admit a natural compactification:*

$$\begin{array}{ccc} \overline{\mathscr{X}} & \hookleftarrow & \mathscr{X} \\ \downarrow & \lrcorner & \downarrow \\ \overline{\mathscr{M}} & \hookleftarrow & \mathscr{M} \end{array}$$

1 Quantum toric stacks

1.1 Step 1 : Replace the tori by quantum tori

We want to replace

$$\mathbb{T}^d := (\mathbb{C}^*)^d = \mathbb{C}^d/\mathbb{Z}^d$$

by $\mathbb{C}^d/\Gamma$ with $\Gamma \subset \mathbb{R}^d$.

Problem : $\mathbb{C}^d/\Gamma$ is not a variety if Γ is not discrete $\rightsquigarrow$ (Analytic) Stacks

Moduli spaces : need to fix the number of generators

Definition 1.1.
- The quantum torus associated to the group epimorphism (or calibration) $h : \mathbb{Z}^n \to \Gamma \subset \mathbb{R}^d$ (such that $h_{|\mathbb{Z}^d} = id$) is the Picard stack ("group stack")

$$\mathscr{T}_h := [\mathbb{C}^d/_h\mathbb{Z}^n] \overset{\mathscr{E}}{\simeq} [\mathbb{T}^d/_{Eh}\mathbb{Z}^{n-d}]$$

- A morphism of quantum tori is a pair of morphisms $(L : \mathbb{R}^d \to \mathbb{R}^{d'}, H : \mathbb{Z}^n \to \mathbb{Z}^{n'})$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{H} & \mathbb{Z}^{n'} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \Gamma & \xrightarrow[L]{} & \Gamma' \end{array}$$

  commutes.
  Equivalently, it is a Picard stack morphism $\mathscr{T}_h \to \mathscr{T}_{h'}$.

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{(x,y)\mapsto(y,2x)} & \mathbb{Z}^2 \\ {\scriptstyle \langle -,(1,\sqrt{2})\rangle}\downarrow & & \downarrow{\scriptstyle \langle -,(1,\sqrt{2})\rangle} \\ \mathbb{Z}+\sqrt{2}\mathbb{Z} & \xrightarrow{z\mapsto z\sqrt{2}} & \mathbb{Z}+\sqrt{2}\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{z\mapsto z\sqrt{2}} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathscr{T}_h = [\mathbb{C}^*/\mathbb{Z}] & \xrightarrow{«z\mapsto z^{\sqrt{2}}»} & \mathscr{T}_h = [\mathbb{C}^*/\mathbb{Z}] \end{array}$$

The obtained morphism has no lifting $\mathbb{T} \to \mathbb{T}$.
The group of morphisms $\mathscr{T}_h \to \mathscr{T}_h$ is $\mathbb{Z} + \sqrt{2}\mathbb{Z}$. More generally, the group of characters of $\mathscr{T}_h$ is

$$\mathrm{im}(h)^* = \{g \in \mathbb{R}^d \mid \forall y \in \mathrm{im}(h)\ \langle g, y\rangle \in \mathbb{Z}\}$$

and the group of cocharacters is isomorphic to $\mathbb{Z}^n$.

1.2 Step 2 : Affine Charts

Definition 1.2. A cone $\sigma \subset \mathbb{R}^d$ is simplicial if it is generated by a $\mathbb{R}$-linear family of $\mathbb{R}^d$ i.e. if there exists an automorphism $L \in \mathrm{GL}_d(\mathbb{R})$ such that $L\sigma = \mathrm{Cone}(e_1, \ldots, e_{\dim(\sigma)})$.

Definition 1.3.
- Let $\sigma \overset{L}{\simeq} \mathrm{Cone}(e_1, \ldots, e_k) \subset \mathbb{R}^d$ be a simplicial cone, $h : \mathbb{Z}^n \to \Gamma \subset \mathbb{R}^d$ be a group epimorphism and $H : \mathbb{R}^n \to \mathbb{R}^n$ an isomorphism such that $(LhH^{-1})_{|\mathbb{Z}^d} = id$ then

$$\mathscr{U}_\sigma := [\mathbb{C}^k \times \mathbb{T}^{d-k} /_{ELhH^{-1}} \mathbb{Z}^{n-d}]$$

- A toric morphism $\mathscr{U}_\sigma \to \mathscr{U}_{\sigma'}$ is a stack morphism which restricts to a torus morphism $\mathscr{T}_h \to \mathscr{T}_{h'}$.

Proposition 1.4. *The correspondance $\sigma \in$ **SimpCones** $\mapsto \mathscr{U}_\sigma \in$ **AffQTS** is an equivalence of categories.*

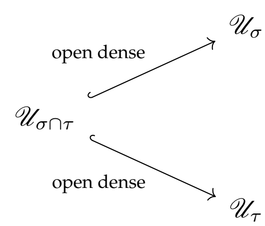
1.3 Step 3 : Gluing

Definition 1.5. A quantum fan is the data of

- an epimorphism $h : \mathbb{Z}^n \to \Gamma$;
- a fan Δ where the rays are generated by the $h(e_i)$, $i = 1 \dots n$.

The elements of the set $\mathscr{I} := [\![1, n]\!] \setminus \Delta(1)$ are called virtual generators.

Let (Δ, h) be a quantum fan. For each cone $\sigma, \tau \in \Delta$, we have



Definition 1.6. The quantum toric stack $\mathscr{X}_{\Delta,h}$ associated to (Δ, h) is the colimit of these diagrams i.e. the gluing of local models

1.4 Main statements on quantum toric stacks

Theorem 1.7 (Katzarkov, Lupercio, Meersseman, Verjovsky, 2020)**.** *The correspondance $(\Delta, h) \in \mathbf{SimpQFans} \mapsto \mathscr{X}_{\Delta,h} \in \mathbf{SimpQTS}$ is an equivalence of categories.*

Theorem 1.8 (Quantum GIT, Katzarkov, Lupercio, Meersseman, Verjovsky, 2020)**.** *If (Δ, h) is a simplicial quantum fan,*

$$\mathscr{X}_{\Delta,h} = [\mathscr{S}(\Delta)/\mathbb{C}^{n-d}]$$

where

- *$\mathscr{S}(\Delta)$ is a quasi-affine (classical) toric variety given by the combinatorics of Δ ;*
- *$\mathbb{C}^{n-d}$ acts on $\mathscr{S}$ through (the exponential of) a Gale transform of h.*

A Gale transform of h is a morphism $k : \mathbb{R}^{n-d} \to \mathbb{R}^n$ such that

$$0 \longrightarrow \mathbb{R}^{n-d} \xrightarrow{k} \mathbb{R}^n \xrightarrow{h} \mathbb{R}^d \longrightarrow 0$$

is exact

Example 1.9. For the quantum projective plane i.e. with $h : (x, y, z) \in \mathbb{Z}^3 \mapsto (x + az, y + bz) \in \Gamma$, with $a, b < 0$ we have:

- $\mathscr{S}(\Delta) = \mathbb{C}^3 \setminus \{0\}$;

- The action of $\mathbb{C}$ on $\mathscr{S}(\Delta)$ is defined by

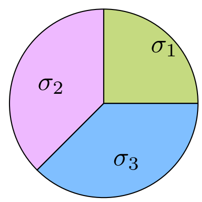
$$t \cdot (z_1, z_2, z_3) = (\exp(2i\pi a t)z_1, \exp(2i\pi b t)z_2, \exp(2i\pi t)z_3).$$

Rationnal case : if $(a, b) \in \mathbb{Q}_{<0}$ then it is a $\mathbb{Z}$-gerbe over a weighted projective plane (if $-(a, b) \in \mathbb{N}$ then it is over $\mathbb{P}(-a, -b, 1)$ and if $(a, b) = -(\frac{p}{q_1}, \frac{r}{q_2})$ then it is over $\mathbb{P}(\frac{pq}{q_1}, \frac{rq}{q_2}, q)$ where $q = q_1 \vee v_2$).

2 Moduli spaces

2.1 Combinatorial type

Definition 2.1. The combinatorial type of a fan is the poset of its cones ordered by inclusion



$S_2 := \mathrm{comb}(\Delta_{\mathbb{P}^2}) = \{1, 2, 3, (1, 2), (1, 3), (2, 3)\}$

Definition 2.2. Let D be the combinatorial type of a fan. A morphism $h : \mathbb{R}^n \to \mathbb{R}^d$ is D-admissible if for all $I \in D$, $\mathrm{Cone}(h(e_i), i \in I)$ is strongly convex.

Example 2.3. $h : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x - z, y - z)$ is S_2-admissible.
$h : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y - z)$ is not S_2-admissible.

2.2 Moduli spaces

Definition 2.4. The moduli space of quantum toric stacks of dimension d, with n generators and of combinatorial type D is

$$\mathscr{M}(d, n, D) = \{h : \mathbb{R}^n \to \mathbb{R}^d \mid h \text{ is } D\text{-admissible}\}/\mathrm{iso}$$

Theorem 2.5 (KLMV,2020 ; B.,2021)**.** *If D is the combinatorial type of a complete simplicial fan, $\mathscr{M}(d, n, D)$ is an orbifold ($\simeq$ a quotient of an open subset $\Omega(d, n, D) \subset \mathbb{R}^{d(n-d)}$[1] by the finite group $\mathrm{Aut}_{\mathbf{Poset}}(D)$)*

The admissibility condition is an open condition thanks to the simpliciality of D.

Theorem 2.6 (B.,2022)**.** *The space $\Omega(d, n, D)$ is a connected semi-algebraic subset of $\mathbb{R}^{d(n-d)}$.*

The space $\Omega(d, n, D)$ is connected thanks to the completeness of D and its inequations are given by determinants describing the combinatorics of D

[1] We fix a cone by automorphism and its d generators

2.3 Examples

Example 2.7. $\Omega(2,3,S_2) = \mathbb{R}^2_{<0}$ and $\mathrm{Aut}(S_2) = D_3 = \mathfrak{S}_3$.
Hence $\mathscr{M}(2,3,S_2)$ has the homotopy type of $B\mathfrak{S}_3 = K(\mathfrak{S}_3, 1)$.
One can compute its singular cohomology with the group cohomology of $\mathfrak{S}_3$.
Its de Rham cohomology is concentrated in degree 0.

More generally, for $S_d = \mathrm{comb}(\Delta_{\mathbb{P}^d})$, we have:
$\Omega(d, d+1, S_d) = \mathbb{R}^d_{<0}$, $\mathrm{Aut}(S_d) = \mathfrak{S}_{d+1}$ and $\mathscr{M}(d, d+1, S_d) \sim B\mathfrak{S}_{d+1} = K(\mathfrak{S}_{d+1}, 1)$.

Proposition 2.8 (B., 2022). *If $d = 2$ then $\Omega(2, n, D)$ is contractible and $\mathscr{M}(2, n, D)$ has the homotopy type of $K(D_n, 1)$*

Example 2.9. The space $\Omega(2, 4, D)$ of (quantum) Hirzebruch surfaces is a fibration of solid hyperbolae over $\mathbb{R}_{<0} \times \mathbb{R}_{<0}$.

2.4 Universal family

Theorem 2.10 (B., 2021). *Let D be the combinatorial type. Then there exists a universal family $\mathscr{X} \to \mathscr{M}(d, n, D)$ of quantum toric stacks of combinatorial type D.*

Sketch of proof. We have a family of quantum GIT :

$$\widetilde{\mathscr{X}} \coloneqq [\mathscr{S}(D) \times \Omega(d, n, D)/\mathbb{C}^{n-d}] \to \Omega(d, n, D)$$

It induces a projection $\mathscr{X} = \widetilde{\mathscr{X}}/\mathrm{Aut}(D) \to \mathscr{M}(d, n, D)$. The tedious point to check is the compatibility of the actions. □

3 Compactification

3.1 Embedding

The morphism

$$\Omega(d, n, D) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^d)^{epi} \xrightarrow{\ker(-)} \mathrm{Gr}(n-d, \mathbb{R}^n)$$

is an open immersion.

Two advantages :

- $\mathrm{Gr}(n-d, \mathbb{R}^n)$ is a compact manifold ;
- The action of $\mathrm{Aut}(D)$ on the image of $\Omega(d, n, D)$ is easier to describe

3.2 Compactification

Theorem 3.1 (B. ; 2021)**.** *There exists a natural compactification $\overline{\mathscr{M}}$ of $\mathscr{M} = \mathscr{M}(d, n, D)$ i.e. there exists a family $\overline{\mathscr{X}} \to \overline{\mathscr{M}}$ such that :*

1. *We have the following commutative diagram:*

$$\begin{array}{ccccc} & & \overline{\mathscr{X}} & \hookleftarrow & \mathscr{X} \\ & & \downarrow & & \downarrow \\ [\mathrm{Gr}(n-d, \mathbb{R}^n)/\mathrm{Aut}(D)] & \hookleftarrow & \overline{\mathscr{M}} & \hookleftarrow & \mathscr{M} \end{array}$$

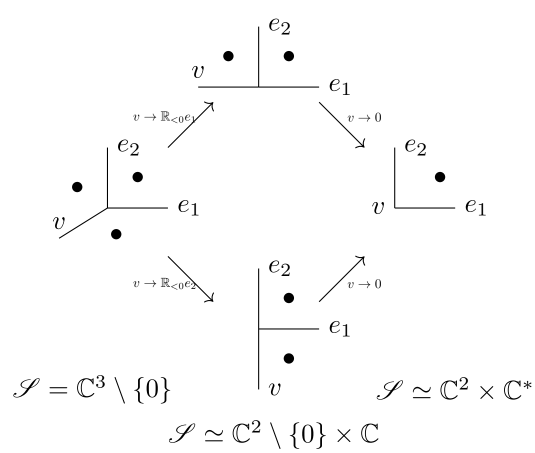
2. *Over a point of $\overline{\mathscr{M}} \setminus \mathscr{M}$, we get a quantum toric stack with a degenerated combinatorial type*[2] *of D (i.e. a subposet of D with the same 1-cones, stable by intersection and taking faces)*

3.3 Example of the moduli space of projective planes

$$\overline{\mathscr{M}(2, 3, D_{\mathbb{P}^2})} = [\mathrm{Conv}([1,0,0],[0,1,0],[0,0,1])/\mathfrak{S}_3] \subset [\mathbb{RP}^2/\mathfrak{S}_3]$$

i.e. a triangle in $\mathbb{RP}^2$ with an action of $\mathfrak{S}_3$ which permutes the vertices and the edges.

On each edges, we get a quotient of $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$ and on each vertices, we get a quotient of $\mathbb{C}^2 \times \mathbb{C}^*$.



[2] In other words, we remove the non-strongly convex cones