Non-simplicial quantum toric varieties

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- Preprint "Non-simplicial quantum toric varieties"
- based on the preprint "Quantum (Non-commutative) Toric Geometry: Foundations" of L.Katzarkov, E.Lupercio, L.Meersseman, A.Verjovsky
  - Goal : Define in a functorial way a "toric variety" associated to a simplicial fan on a finitely generated subgroup of $\mathbb{R}^d$ (quasi-lattice)
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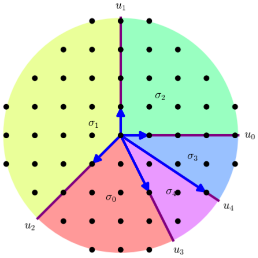
- In classical theory, simplicial fans correspond to the orbifold toric varieties

Non-simplicial cones I

- exist in dimension ≥ 3



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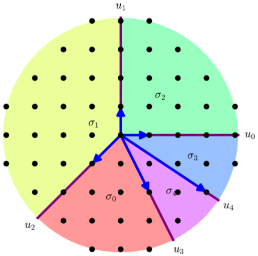
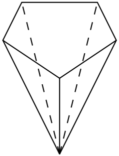


- can have an arbitrary number of 1-cones



Non-simplicial cones II

- don't have a nice description of their faces

Non-simplicial cones II

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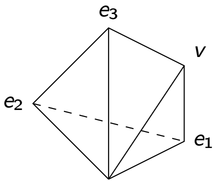
  $\sigma = \mathrm{Cone}(e_1, e_2, e_3, v := e_1 - e_2 + e_3) \subset \mathbb{R}^3$



Its faces are :

- Cone(e_1), Cone(e_2),Cone(e_3), Cone(v) (with 1 generator) ;
- Cone(e_1, e_2), Cone(e_2, e_3), Cone(e_1, v), Cone(e_3, v) (with 2 generators) ;
- no cones with 3 generators

Stacks

Site $\mathfrak{A}$:

- Objects: affine toric varieties
- Morphisms : toric morphisms
- Coverings : $\{U_i \hookrightarrow X\}_{i\in\{1,\dots,n\}}$ where the U_i are toric open subsets of X such that $X = \bigcup_{i=1}^{n} U_i$

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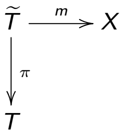
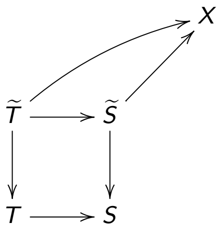
Stacks over $\mathfrak{A}$:
Let H be an abelian Lie group acting on a toric variety X. We can consider the quotient stack $[X/H]$:

- objects over an object $T \in \mathfrak{A}$:



(where π is a H-principal bundle and m is H-equivariant)

- morphisms over a morphism $T \to S$



First definitions of Quantum Toric Geometry

- Calibration of $\Gamma \subset \mathbb{R}^d$ ($\mathbb{Z}^d \subset \Gamma$) :
  - an epimorphism $h : \mathbb{Z}^N \to \Gamma$ such that $h(e_i) = e_i$ for $i \in \{1, \ldots, d\}$;
  - A subset $\mathcal{I} \subset \{1, \ldots, N\}$ such that $\mathrm{Vect}_{\mathbb{C}}(h(e_i), i \notin \mathcal{I}) = \mathbb{C}^d$

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- Quantum Torus associated to a calibration $(h, \mathcal{I})$: the quotient stack

$$\mathscr{T}^{cal}_{h,\mathcal{I}} := [\mathbb{C}^d / \mathbb{Z}^N]$$

where the action of $\mathbb{Z}^N$ on $\mathbb{C}^d$ is

$$m \cdot z = z + h(m)$$

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First definitions of Quantum Toric Geometry II

- A morphism of quantum tori is given by two linear morphisms $L : \mathbb{R}^d \to \mathbb{R}^{d'}$ and $H : \mathbb{R}^N \to \mathbb{R}^{N'}$ with a map $s : \mathcal{I} \to \mathcal{I}'$, compatible with the calibrations

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- Multiplicative form of the quantum torus :

$$\mathscr{T}_{h,\mathcal{I}}^{cal} \simeq \left[\mathbb{T}^d / \mathbb{Z}^{N-d}\right]$$

where the action of $\mathbb{Z}^{N-d}$ on $\mathbb{T}^d := (\mathbb{C}^*)^d$ is

$$m \cdot z = E(h(0_{\mathbb{Z}^d} \oplus m))z$$

where $E(z_1, \ldots, z_d) = (\exp(2i\pi z_1), \ldots, \exp(2i\pi z_d))$

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Presented calibrated quantum tori I

Definition

A presented calibrated quantum torus is a 6-uple

$$(\mathscr{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G \subset \mathbb{R}^p, \mathcal{I}', L, H, s)$$

where $(\varphi, \mathcal{I}')$ is a calibration of the group G, $L : \mathbb{R}^p \to \mathbb{R}^d$ is a linear epimorphism, $H : \mathbb{Z}^N \to \mathbb{Z}^N$ is a group isomorphism and $s : \mathcal{I} \to \mathcal{I}'$ is a bijection such that :

- $L_{|G} : G \to \Gamma$ is a group isomorphism.
- $hH = L\varphi$
- For all $i \in \mathcal{I}$, $H(e_i) = e_{s(i)}$ and for all $i \notin \mathcal{I}$, $H(e_i) \in \bigoplus_{j \notin \mathcal{I}'} \mathbb{Z}e_j$

The morphism φ is the calibration of this presented calibrated quantum torus.

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- If $p = d$, this data define a torus isomorphism between $\mathscr{T}^{cal}_{h,\mathcal{I}}$ and $\mathscr{T}^{cal}_{\varphi,\mathcal{I}'}$
- In the general case, they define a stack isomorphism between the associated stack $[\mathbb{C}^p/\mathbb{Z}^N \times \ker(L \otimes_{\mathbb{R}} id_{\mathbb{C}})]$ and $\mathscr{T}^{cal}_{h,\mathcal{I}}$

Presented calibrated quantum tori II

- Morphism between presented calibrated quantum tori = torus morphism $\mathscr{L}^{cal} : \mathscr{T}^{cal}_{h,\mathcal{I}} \to \mathscr{T}^{cal}_{h',\mathcal{I}'}$ with morphisms compatible with the data of the presentation.
- The following diagram commutes

$$\begin{array}{ccc} [\mathbb{C}^p/\mathbb{Z}^N \times \ker(L_{\mathbb{C}})] & \xrightarrow{\mathscr{L}'} & [\mathbb{C}^{p'}/\mathbb{Z}^{N'} \times \ker(L'_{\mathbb{C}})] \\ \downarrow \simeq & & \downarrow \simeq \\ \mathscr{T}^{cal}_{h,\mathcal{I}} & \xrightarrow{\mathscr{L}^{cal}} & \mathscr{T}^{cal}_{h',\mathcal{I}'} \end{array}$$

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- There exists a multiplicative form $[\mathbb{T}^p/\mathbb{Z}^{N-d} \times E(\ker(L \otimes_{\mathbb{R}} id_{\mathbb{C}}))]$

Forget the presentation

Theorem (B. ; 2020)

The forgetful functor

$$(\mathscr{T}^{cal}_{h,\mathcal{I}}, \varphi : \mathbb{Z}^N \to G, \mathcal{I}', L, H, s) \to \mathscr{T}^{cal}_{h,\mathcal{I}},$$

is an equivalence of categories between the category of presented calibrated quantum tori and the category of standard calibrated quantum tori.

Setting

- A finitely generated subgroup Γ of $\mathbb{R}^d$;
- A calibration $(h^{cal} : \mathbb{Z}^N \to \Gamma, \mathcal{I})$ of Γ ;
- A strongly convex cone $\sigma = \sigma_I$ of $\mathbb{R}^d$ of dimension d (i.e. $\dim \mathrm{Vect}(\sigma) = d$) which is generated by some $v_i := h^{cal}(e_i), i \in I \subset \{1, \dots, N\} \setminus \mathcal{I}$.

Construction of quantum toric varieties I

With these data, we can consider

- $h_{\sigma\mathbb{C}} := h^{cal}_{\mathbb{C}|\mathbb{C}^I} : \mathbb{C}^I \to \mathbb{C}^d$ which is an epimorphism ;
- a basis $\mathcal{B} = \left\{ v_j, j \in \widetilde{I} \right\}$ of $\mathbb{C}^d$ included in the set of the generators of the 1-cones of σ which induces a decomposition

$$\mathbb{C}^I = \mathbb{C}^{\widetilde{I}} \oplus \ker(h_{\sigma\mathbb{C}});$$

- a permutation $\chi \in \mathfrak{S}_N$ such that $\chi(\{1, \ldots, d\}) = \widetilde{I}$

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Definition

The linear morphism $\varphi : \mathbb{C}^N \to \mathbb{C}^{\widetilde{I}} \hookrightarrow \mathbb{C}^I$ defined by

$$e_k \mapsto [h_{\sigma\mathbb{C}}]^{-1}(h^{cal}_{\mathbb{C}}(e_{\chi(k)}))$$

is called calibration associated to σ, $\mathcal{B}$ and χ.

Construction of quantum toric varieties II

By analogy with the tori, we can now define an action of $\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))$ on $\mathbb{C}^I$ by :

$$(m, E(t)) \cdot z = E(\varphi(0 \oplus m) + t)z$$

and define the quotient stack associated to it :

$$\mathscr{U}_\sigma^{cal} := [\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$$

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Definition

The stack $\mathscr{U}_\sigma^{cal} := [\mathbb{C}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$ is the quantum toric variety associated to the cone σ and to the calibration $h^{cal} : \mathbb{Z}^N \to \Gamma$.

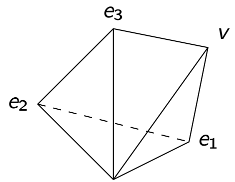
The 6-uple

$$(\mathscr{T}^{cal}_{h^{cal},\mathcal{I}}, \varphi : \mathbb{Z}^N \to [h_\sigma]^{-1}(\Gamma), \chi^{-1}(\mathcal{I}), h_{\sigma\mathbb{C}}, P_\chi : e_i \mapsto e_{\chi(i)}, \chi)$$

is the presented quantum torus associated to $\mathscr{U}_\sigma^{cal}$ which encodes the stack $[\mathbb{T}^I / \mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))]$.

Example

Let $h^{cal} : \mathbb{Z}^N \to \mathbb{R}^3$, $e_i \mapsto e_i$ for $i = 1, 2, 3$, $e_4 \mapsto v := ae_1 - be_2 + ce_3$, $e_k \mapsto v_k$ and $\Gamma = h^{cal}(\mathbb{Z}^N)$. Let $\sigma = \text{Cone}(e_1, e_2, e_3, v)$.



This cone encodes an action of $\mathbb{Z}^{N-3} \times E(\mathbb{C}(v, -1))$ on $\mathbb{C}^4 = (\mathbb{C}^3 \times 0) \oplus \mathbb{C}(v, -1)$ defined by :

$$(m, E(\lambda(v, -1)) \cdot z = E(h^{cal}(m) + \lambda(v, -1))z$$

The quantum toric variety associated to σ is the quotient stack

$$\mathscr{U}_\sigma^{cal} = [\mathbb{C}^4 / \mathbb{Z}^{N-3} \times E(\mathbb{C}(v, -1))]$$

Links with the other constructions

- Consider the classical setting i.e. Γ is a lattice and the calibration is an isomorphism, then
    - $\mathscr{U}_\sigma^{cal}$ is not isomorphic to the toric variety U_σ
    - $U_\sigma = \mathbb{C}^I \mathbin{/\!/} E(\ker(h_{\sigma\mathbb{C}}))$

Links with the other constructions

- Consider the classical setting i.e. Γ is a lattice and the calibration is an isomorphism, then
  - $\mathscr{U}_\sigma^{cal}$ is not isomorphic to the toric variety U_σ
  - $U_\sigma = \mathbb{C}^l \mathbin{/\!/} E(\ker(h_{\sigma\mathbb{C}}))$
- For a general Γ but with σ simplicial, we have a toric isomorphism :

$$[\mathbb{C}^l/\mathbb{Z}^{N-d} \times E(\ker(h_{\sigma\mathbb{C}}))] = \mathscr{U}_\sigma^{cal} \simeq Q^{cal}_{d,P_\chi^{-1}\circ\varphi} := [\mathbb{C}^d/\mathbb{Z}^{N-d}]$$

Compatibility with the restriction

If $\dim \tau = k < d$, we can choose a family J of elements of $\{1, \dots, N\}$ of cardinal $d - k$ such that $\mathrm{Vect}(\tau \cup v_j, j \in J) = \mathbb{C}^d$.
Then, by the same construction,

$$\mathscr{U}_\tau^{cal} := \left[\mathbb{C}^I \times \mathbb{T}^J / \mathbb{Z}^{N-d} \times E \left(\ker \left(h^{cal}_{\mathbb{C}|\mathbb{C}^I \times \mathbb{C}^J} \right) \right) \right]$$

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Proposition

Let $\sigma = \sigma_I$ be a cone and let $\tau = \sigma_{I'}$ be a face of σ. Then we have an isomorphism

$$\mathscr{U}_\tau^{cal} \simeq [\mathbb{C}^{I'} \times \mathbb{T}^{I \setminus I'} / \mathbb{Z}^{N-d} \times E(\ker(h_\sigma))] \hookrightarrow \mathscr{U}_\sigma^{cal}$$

which restricts to an torus isomorphism.

Setting II

- A finitely generated subgroup Γ of $\mathbb{R}^d$;
- A calibration $(h^{cal} : \mathbb{Z}^N \to \Gamma, \mathcal{I})$ of Γ ;
- A family of strongly convex cones $\sigma = \sigma_I$ of $\mathbb{R}^d$ which are generated by some $v_i := h^{cal}(e_i), i \in I \subset \{1, \ldots, N\} \setminus \mathcal{I}$ such that
  - every intersection of cones is a cone ;
  - every face of a cone is a cone

Quantum toric varieties

Definition

Let $T \in \mathfrak{A}$. An object of $\mathscr{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}}$ over T is a covering $(T_I := T_{\sigma_I})_{I \in \Delta_{max}}$ of T together with an object of $\mathscr{U}^{cal}_{\sigma_I}$ over T_I

$$\begin{array}{ccc} \widetilde{T}_I & \xrightarrow{m_I} & \mathbb{C}^I \times \mathbb{T}^K \\ \downarrow & & \\ T_I & & \end{array}$$

for every $\sigma_I \in \Delta_{max}$, satisfying for any couple (I, I') with non-empty intersection J

$$\mathscr{G}_{II'} \left(\begin{array}{ccc} m_I^{-1}(\mathscr{S}_{\sigma_I \sigma_{I'}}) & \xrightarrow{m_I} & \mathscr{S}_{\sigma_I \sigma_{I'}} \\ \downarrow & & \\ T_I & & \end{array} \right) = \begin{array}{ccc} m_{I'}^{-1}(\mathscr{S}_{\sigma_{I'} \sigma_I}) & \xrightarrow{m_{I'}} & \mathscr{S}_{\sigma_{I'} \sigma_I} \\ \downarrow & & \\ T_{I'} & & \end{array}$$

Correspondence

Theorem (Katzarkov, Lupercio, Meersseman, Verjovsky ; 2020)

The correspondence $(\Delta, h^{cal}, \mathcal{I}) \mapsto \mathscr{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}}$ is functorial and induces an equivalence of categories between the category of simplicial calibrated quantum fans and the category of simplicial calibrated quantum toric varieties.

Theorem (B. ; 2020)

The correspondence $(\Delta, h^{cal}, \mathcal{I}) \mapsto \mathscr{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}}$ is functorial and induces an equivalence of categories between the category of calibrated quantum fans and the category of calibrated quantum toric varieties.

Global quotient

Let $(\Delta, h^{cal} : \mathbb{Z}^N \to \Gamma, \mathcal{I})$ be a quantum fan.
The associated fan $\Delta_{h^{cal}}$ is the fan on $\mathbb{Z}^N$ whose maximal fans are the cones $\mathrm{Cone}(e_i, i \in I)$ where $\sigma_I \in \Delta_{max}$.
Let $\mathscr{S} = X(\Delta_{h^{cal}})$ the (classical) toric variety associated to this fan.

Theorem

There is a stack isomorphism

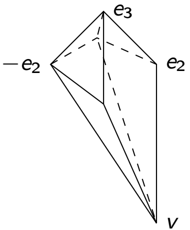
$$\mathscr{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}} \simeq [\mathscr{S} / \mathbb{Z}^{N-d} \times E(\ker(h^{cal}_{\mathbb{C}}))]$$

which restricts to a torus isomorphism between the associated quantum torus on each affine chart.

Example II

Let $\varepsilon \in \mathbb{R}_{>0}$, $v = \left(-\frac{1}{\varepsilon}, \frac{2+2\varepsilon}{\varepsilon}, -\frac{2}{\varepsilon}\right)$, $\Gamma = \mathbb{Z}^3 + \mathbb{Z}v \subset \mathbb{R}^3$, $h^{cal} : \mathbb{Z}^6 \to \Gamma$ be a standard calibration of Γ, Δ the fan of $\mathbb{R}^3$ whose maximal fans are

$$\Delta_{max} = \{\mathrm{Cone}(e_1, \pm e_2, e_3), \mathrm{Cone}(-e_1, -e_2, e_3), \mathrm{Cone}(e_1, \pm e_2, v), \mathrm{Cone}(-e_1, -e_2, v), \mathrm{Cone}(-e_1, e_2, e_3, v)\}$$



$\mathscr{X}^{cal}_{\Delta, h^{cal}, \emptyset}$ is isomorphic to the quotient of

$$\mathscr{S} = (\mathbb{C}^2 \setminus \{0\})^3 \setminus [\mathbb{C}^* \times \mathbb{C}^3 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^* \times \mathbb{C})^3] \cup (\mathbb{C}^* \times \mathbb{C}^3 \times \mathbb{C}^* \times \mathbb{C})$$

by the action of $\mathbb{Z}^{N-d} \times E(\ker(h^{cal}_{\mathbb{C}}))$ defined by

$$(m, E(t)) \cdot z = E(h^{cal}(m) \oplus 0_{\mathbb{C}^{N-d}} + t)z$$

Quantum GIT quotient

A Gale transform of a family $\{v_1, \ldots, v_N\} \subset \mathbb{R}^d$ is a family $\{A_1, \ldots, A_N\} \subset \mathbb{R}^{N-d}$ such that the morphisms $h\colon (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i v_i$ and $k\colon t \in \mathbb{R}^{n-d} \mapsto (\langle A_1, t\rangle, \ldots, \langle A_N, t\rangle) \in \mathbb{R}^N$ make the following sequence exact

$$0 \longrightarrow \mathbb{R}^{N-d} \xrightarrow{k} \mathbb{R}^N \xrightarrow{h} \mathbb{R}^d \longrightarrow 0$$

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Theorem (Katzarkov, Lupercio, Meersseman, Verjovsky ; 2020)

If the quantum fan is simplicial, the quantum toric variety $\mathscr{X}^{cal}_{\Delta, h^{cal}, \mathcal{I}}$ can be described as a global quotient $[\mathscr{S}/\mathbb{C}^{N-d}]$ where $\mathbb{C}^{N-d}$ acts on $\mathscr{S}$ through the morphism $E \circ k_{\mathbb{C}} : \mathbb{C}^{N-d} \to \mathbb{T}^N$ where k is defined by a Gale transform of the family $(h^{cal}(e_i))_{i\in\{1,\dots,N\}}$

Idea of proof :
We can prove that the open substack $\mathscr{U}'_{\sigma_I} := [\mathbb{C}^I \times \mathbb{T}^{I^c}/\mathbb{C}^{N-d}] \subset [\mathscr{S}/\mathbb{C}^{N-d}]$ and $\mathscr{U}^{cal}_{\sigma_I}$ are isomorphic.

Proposition (B. ; 2020)

The stacks $\mathscr{U}_\sigma^{cal}$ and $\mathscr{U}'_\sigma$ are not isomorphic if σ is not simplicial.

Hence, if Δ is not simplicial, $\mathscr{X}^{cal}_{\Delta,h^{cal},\mathcal{I}}$ and $[\mathscr{S}/\mathbb{C}^{N-d}]$ are not isomorphic