Toric varieties and beyond

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Definition

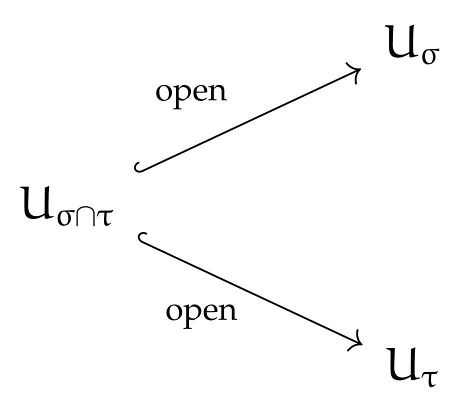
A toric variety is a normal complex variety with an action of the torus $(\mathbb{C}^*)^d$ having a dense orbit.

A combinatorial construction of toric varieties

In order to build toric varieties, we will consider fans in $\mathbb{R}^d$ i.e. families of rational (i.e. which have a family of generators in $\mathbb{Z}^d$) strongly convex (i.e. which do not contain lines) cones stable under taking intersections and faces.
To each cone σ in a fan Σ, we associate the affine toric variety

$$U_\sigma := \mathrm{Spm}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^d])$$

where $\sigma^\vee := \{x \in \mathbb{R}^d \mid \forall u \in \sigma, \langle u, x\rangle \geqslant 0\}$ is the dual cone of σ.
If we take two cones σ and τ in Σ then their intersection $\sigma \cap \tau$ is a (non-empty) cone in Σ and we have the following diagram



For each couple (σ, τ), it defines transition maps between U_σ and U_τ Thus, they allow us to glue the affine varieties U_σ into a variety, which is also toric, denoted by X_Σ.

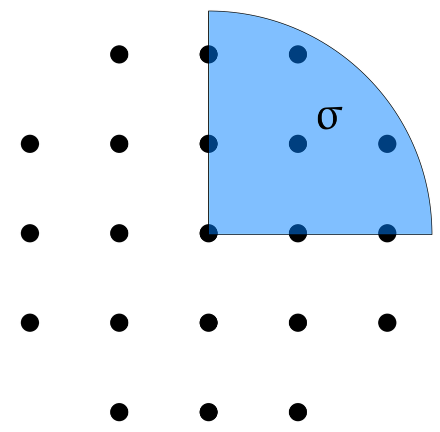
Central theorem

Theorem *The map $\Sigma \mapsto X_\Sigma$ induces an equivalence of categories*

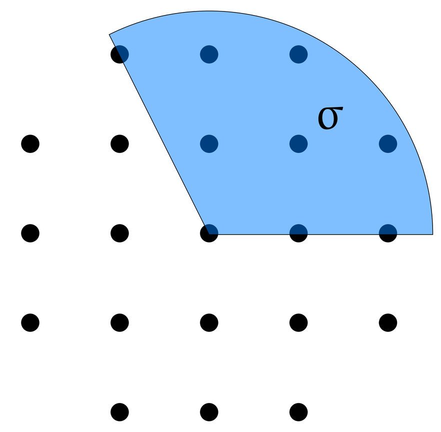
$$\mathfrak{Fans} \simeq \mathfrak{ToricVar}$$

Examples

Affine toric varieties

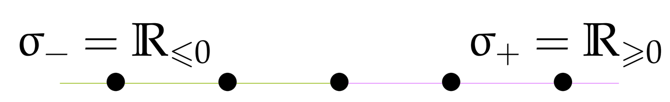


$\mathrm{Spm}(\mathbb{C}[x, y]) = \mathbb{C}^2$

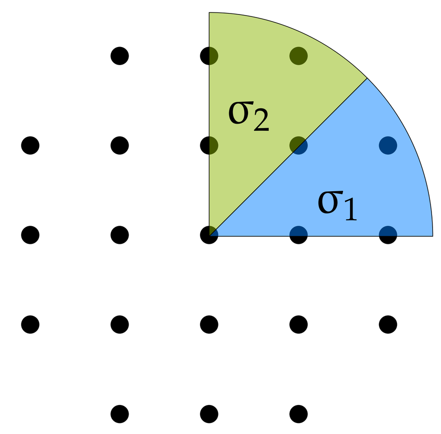


$\mathrm{Spm}(\mathbb{C}[y, xy, x^2y]) \simeq \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$

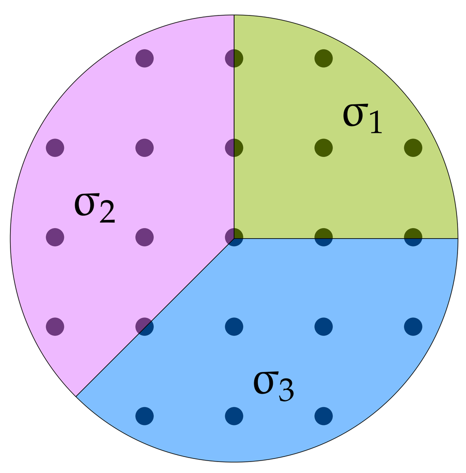
General toric varieties



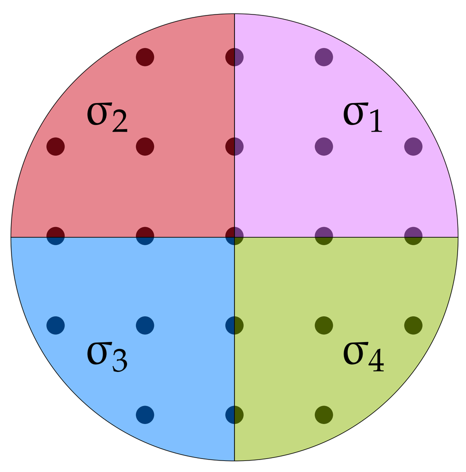
$\mathbb{P}^1$



$\mathrm{Bl}_0\mathbb{C}^2$



$\mathbb{P}^2$



$\mathbb{P}^1 \times \mathbb{P}^1$

Geometry-combinatorics dictionary

Geometry de X_Σ	Combinatorics of Σ
A $(\mathbb{C}^*)^d$-orbit $O(\sigma) := \mathrm{Spm}(\mathbb{C}[\sigma^\perp \cap \mathbb{Z}^d]) \simeq (\mathbb{C}^*)^{d-\dim\sigma}$ of X_Σ	A cone σ of Σ
X_Σ is a smooth variety	Each cone of Σ is smooth (i.e. generated by a sub-family of a basis of $\mathbb{Z}^d$) In general, $\mathrm{Reg}(X_\Sigma) = \bigcup_{\sigma \text{ smooth}} U_\sigma$
X_Σ is an orbifold (locally the quotient of an open subset of $\mathbb{C}^d$ by a finite group)	Each cone of Σ is generated by a linearly independent family in $\mathbb{R}^d$
X_Σ is a proper variety (i.e. compact for the Euclidean topology)	$\lvert\Sigma\rvert := \bigcup_{\sigma\in\Sigma} \sigma = \mathbb{R}^d$
X_Σ is a projective variety	The maximal cones are generated by the faces of a polytope P (defining an ample divisor D_P)
Blow up along the (equivariant) divisor $\overline{O(\sigma)}$	Barycentric subdivision of σ

Towards quantum toric geometry

Problem *Since the cones of a fan of a toric variety are rationals, we can not continuously deform a toric variety by deforming the cones of its fan:*
If we continuously deform the cone $\mathrm{Cone}((-1,-1)) = \mathbb{R}_{\geqslant 0}(-1,-1)$ *of the fan of* $\mathbb{P}^2$ *(the other rays remain unchanged), we might get cones of the form* $\mathrm{Cone}(\alpha)$ *where* $\alpha \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. *In these cases, the obtained cones can not be rational (since the rank of the group* $\Gamma = \mathbb{Z}^2 + \alpha\mathbb{Z}$ *is 3).*

Solution *Replace classical objects by "quantum" ones:*

	Classical	*Quantum*
Studied objects	*Orbifold varieties*	*Analytic stacks*
Tori	$(\mathbb{C}^*)^d = \mathbb{C}^d/\mathbb{Z}^d$	$[\mathbb{C}^d/\Gamma] = [(\mathbb{C}^*)^d/\exp(2i\pi\Gamma)]$ *where* Γ *is a finitely generated subgroup of* $\mathbb{R}^d$.
Cones	*Simplicial rational cones*	*Simplicial cones generated by elements of* Γ
Affine charts	$\mathbb{C}^d/G$ *with finite* G	$[\mathbb{C}^d/\exp(2i\pi\Gamma)]$

References

[1] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011.

[2] L. Katzarkov, E. Lupercio, L. Meersseman, and A. Verjovsky. Quantum (non-commutative) toric geometry: Foundations. *Advances in Mathematics*, 391:107945, 11 2021.