Compactification of moduli spaces in quantum toric geometry

Antoine Boivin
Ph.D. thesis under the supervision of Laurent Meersseman

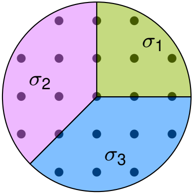
Laboratoire Angevin de REcherche en MAthématiques

November 30, 2022

Classical toric varieties

Toric variety = normal complex variety with an action of an algebraic torus $(\mathbb{C}^*)^d$ having a dense orbit

Main interest : Fully described by fans i.e. families of rational strongly convex cones (stable by intersection and taking faces).



Theorem

This correspondance is an equivalence of categories **Fans** $\rightarrow$ **Torics**

Historical background

1970 : Introduced by Michel DEMAZURE in order to study the Cremona group.

1993 : Victor BATYREV proves that hypersurfaces of certain toric varieties supply examples of mirror symmetry.

2000s : development of the theory of toric stacks (L. BORISOV, L. CHEN, G. SMITH, B. FANTECHI, É. MANN, F. NIRONI, A. GERASCHENKO and M. SATRIANO).

2006 : Homological mirror symmetry proved for smooth projective toric varieties by Mohammed ABOUZAID.

2014 : "Non-commutative toric varieties" are introduced by L. KATZARKOV, E. LUPERCIO, L. MEERSSEMAN and A. VERJOVSKY as leaf stack of LVM manifold.

2020 : They extend their construction to the "quantum toric stacks" for simplicial irrational fans.

Moduli spaces and rationality condition

Rationality condition $\Rightarrow$ toric varieties are rigid as equivariant spaces :
The continuous deformation of cones and their underlying lattice leads to dense subgroups of $\mathbb{R}^d$

Example

$$\Gamma_\alpha = \mathbb{Z}^2 + \alpha\mathbb{Z} \begin{cases} \text{is discrete and of rank 2 if } \alpha \in \mathbb{Q}^2 \\ \text{is not discrete and can be dense otherwise} \end{cases}$$

$\rightsquigarrow$ No moduli spaces of toric varieties.

We need to consider more general objets.

Table of Contents

Table of Contents

Step 1 : Replace the tori by quantum tori

We want to replace

$$\mathbb{T}^d := (\mathbb{C}^*)^d = \mathbb{C}^d/\mathbb{Z}^d$$

by $\mathbb{C}^d/\Gamma$ with $\Gamma \subset \mathbb{R}^d$.
Problem : $\mathbb{C}^d/\Gamma$ is not a variety if Γ is not discrete $\rightsquigarrow$ (Analytic) Stacks

Moduli spaces : need to fix the number of generators

Step 1 : Replace the tori by quantum tori

We want to replace

$$\mathbb{T}^d := (\mathbb{C}^*)^d = \mathbb{C}^d/\mathbb{Z}^d$$

by $\mathbb{C}^d/\Gamma$ with $\Gamma \subset \mathbb{R}^d$.
Problem : $\mathbb{C}^d/\Gamma$ is not a variety if Γ is not discrete $\rightsquigarrow$ (Analytic) Stacks

Moduli spaces : need to fix the number of generators

Definition

- The quantum torus associated to the group epimorphism (or calibration) $h : \mathbb{Z}^n \to \Gamma \subset \mathbb{R}^d$ is the Picard stack

$$\mathscr{T}_h := [\mathbb{C}^d/_h\mathbb{Z}^n] \overset{\mathscr{E}}{\simeq} [\mathbb{T}^d/_{Eh}\mathbb{Z}^{n-d}]$$

- A morphism of quantum tori is a pair of morphisms $(L : \mathbb{R}^d \to \mathbb{R}^{d'}, H : \mathbb{Z}^n \to \mathbb{Z}^{n'})$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{H} & \mathbb{Z}^{n'} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \Gamma & \xrightarrow[L]{} & \Gamma' \end{array}$$

commutes.
Equivalently, it is a Picard stack morphism $\mathscr{T}_h \to \mathscr{T}_{h'}$.

Example of torus morphism

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{(x,y)\mapsto(y,2x)} & \mathbb{Z}^2 \\ {\scriptstyle \langle -,(1,\sqrt{2})\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle \langle -,(1,\sqrt{2})\rangle} \\ \mathbb{Z}+\sqrt{2}\mathbb{Z} & \xrightarrow{z\mapsto z\sqrt{2}} & \mathbb{Z}+\sqrt{2}\mathbb{Z} \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{C} & \xrightarrow{z\mapsto z\sqrt{2}} & \mathbb{C} \\ \Big\downarrow & & \Big\downarrow \\ \mathscr{T}_h = [\mathbb{C}^*/\mathbb{Z}] & \xrightarrow{\text{«}z\mapsto z^{\sqrt{2}}\text{»}} & \mathscr{T}_h = [\mathbb{C}^*/\mathbb{Z}] \end{array}$$

Step 2 : Affine Charts

Definition

- Let $\sigma \overset{L}{\simeq} \mathrm{Cone}(e_1, \ldots, e_k) \subset \mathbb{R}^d$ be a simplicial cone and $h : \mathbb{Z}^n \to \Gamma \subset \mathbb{R}^d$ be a group epimorphism then
$$\mathscr{U}_\sigma := [\mathbb{C}^k \times \mathbb{T}^{d-k} / _{EL^{-1}h}\mathbb{Z}^{n-d}]$$
- A toric morphism $\mathscr{U}_\sigma \to \mathscr{U}_{\sigma'}$ is a stack morphism which restricts to a torus morphism $\mathscr{T}_h \to \mathscr{T}_{h'}$.

Proposition

*The correspondance $\sigma \in$ **SimpCones** $\mapsto \mathscr{U}_\sigma \in$ **AffQTS** is an equivalence of categories.*

Step 3 : Gluing

Definition

A quantum fan is the data of

- an epimorphism $h : \mathbb{Z}^n \to \Gamma$;
- a fan Δ where the rays are generated by the $h(e_i)$, $i = 1 \dots n$.

The elements of the set $\mathcal{I} := [\![1, n]\!] \setminus \Delta(1)$ are called virtual generators.

Step 3 : Gluing

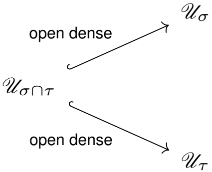
Definition

A quantum fan is the data of

- an epimorphism $h : \mathbb{Z}^n \to \Gamma$;
- a fan Δ where the rays are generated by the $h(e_i)$, $i = 1 \ldots n$.

The elements of the set $\mathscr{I} := [\![1, n]\!] \setminus \Delta(1)$ are called virtual generators.

Let (Δ, h) be a quantum fan. For each cone $\sigma, \tau \in \Delta$, we have



Definition

The quantum toric stack $\mathscr{X}_{\Delta,h}$ associated to (Δ, h) is the colimit of these diagrams.

Main statements on quantum toric stacks

Theorem (Katzarkov, Lupercio, Meersseman, Verjovsky, 2020)

The correspondance $(\Delta, h) \in$ **SimpQFans** $\mapsto \mathscr{X}_{\Delta,h} \in$ **SimpQTS** *is an equivalence of categories.*

Theorem (Quantum GIT, Katzarkov, Lupercio, Meersseman, Verjovsky, 2020)

If (Δ, h) *is a simplicial quantum fan,*

$$\mathscr{X}_{\Delta,h} = [\mathscr{S}(\Delta)/\mathbb{C}^{n-d}]$$

where

- $\mathscr{S}(\Delta)$ *is a quasi-affine (classical) toric variety given by the combinatorics of* Δ *;*
- $\mathbb{C}^{n-d}$ *acts on* $\mathscr{S}$ *through a Gale transform of* h*.*

A Gale transform of h is a morphism $k : \mathbb{R}^{n-d} \to \mathbb{R}^n$ such that

$$0 \longrightarrow \mathbb{R}^{n-d} \xrightarrow{k} \mathbb{R}^n \xrightarrow{h} \mathbb{R}^d \longrightarrow 0$$

is exact

General case

We can extend the two previous approaches (Gluing local models and QGIT) to the general (i.e. not necessarily simplicial) case.

Theorem (B., 2020)

The two constructions lead to an equivalence of categories **QFans** $\to$ **QTS***.*

But

Theorem (B., 2020)

If the considered quantum fan is not simplicial then these two constructions do not coincide.

General case

We can extend the two previous approaches (Gluing local models and QGIT) to the general (i.e. not necessarily simplicial) case.

Theorem (B., 2020)

The two constructions lead to an equivalence of categories **QFans** $\to$ **QTS**.

But

Theorem (B., 2020)

If the considered quantum fan is not simplicial then these two constructions do not coincide.

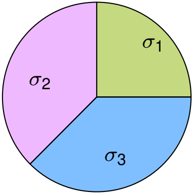
In what follows, we will consider the QGIT one i.e. $\mathscr{X}_{\Delta,h} = [\mathscr{S}(\Delta)/\mathbb{C}^{n-d}]$ in order to keep the same group $\mathbb{C}^{n-d}$ when we will consider families of toric stacks

Table of Contents

Combinatorial type

Definition

The combinatorial type of a fan is the poset of its cones ordered by inclusion



$S_2 \coloneqq \mathrm{comb}(\Delta_{\mathbb{P}^2}) = \{1, 2, 3, (1, 2), (1, 3), (2, 3)\}$

Definition

Let D be the combinatorial type of a fan. A morphism $h : \mathbb{R}^n \to \mathbb{R}^d$ is D-admissible if for all $I \in D$, $\mathrm{Cone}(h(e_i), i \in I)$ is strongly convex.

Example

$h : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x - z, y - z)$ is S_2-admissible.
$h : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y - z)$ is not S_2-admissible.

Definition

The moduli space of quantum toric stacks of dimension d, with n generators and of combinatorial type D is

$$\mathscr{M}(d, n, D) = \{h : \mathbb{R}^n \to \mathbb{R}^d \mid h \text{ is } D\text{-admissible}\}/\mathrm{iso}$$

Moduli spaces

Definition

The moduli space of quantum toric stacks of dimension d, with n generators and of combinatorial type D is

$$\mathscr{M}(d,n,D) = \{h : \mathbb{R}^n \to \mathbb{R}^d \mid h \text{ is } D\text{-admissible}\}/\mathrm{iso}$$

Theorem (KLMV,2020 ; B.,2021)

If D is the combinatorial type of a complete simplicial fan, $\mathscr{M}(d,n,D)$ is an orbifold ($\simeq$ a quotient of an open subset $\Omega(d,n,D) \subset \mathbb{R}^{d(n-d)}$ by the finite group $\mathrm{Aut}_{\mathbf{Poset}}(D)$)

Theorem (B.,2022)

The space $\Omega(d,n,D)$ is a connected semi-algebraic subset of $\mathbb{R}^{d(n-d)}$.

Examples

Example

$\Omega(2,3,S_2) = \mathbb{R}^2_{<0}$ and $\mathrm{Aut}(S_2) = D_3 = \mathfrak{S}_3$.
Hence $\mathscr{M}(2,3,S_2)$ has the homotopy type of $B\mathfrak{S}_3 = K(\mathfrak{S}_3, 1)$.
One can compute its singular cohomology with the group cohomology of $\mathfrak{S}_3$.

More generally, for $S_d = \mathrm{comb}(\Delta_{\mathbb{P}^d})$, we have:
$\Omega(d, d+1, S_d) = \mathbb{R}^d_{<0}$, $\mathrm{Aut}(S_d) = \mathfrak{S}_{d+1}$ and $\mathscr{M}(d, d+1, S_d) \sim B\mathfrak{S}_{d+1} = K(\mathfrak{S}_{d+1}, 1)$.

Proposition (B., 2022)

If $d = 2$ then $\Omega(2, n, D)$ is contractible and $\mathscr{M}(2, n, D)$ has the homotopy type of $K(D_n, 1)$

Example

The space $\Omega(2, 4, D)$ of (quantum) Hirzebruch surfaces is a fibration of solid hyperbolae over $\mathbb{R}_{<0} \times \mathbb{R}_{<0}$.

Universal family

Theorem (B., 2021)

Let D be the combinatorial type. Then there exists a universal family $\mathscr{X} \to \mathscr{M}(d,n,D)$ of quantum toric stacks of combinatorial type D.

Sketch of proof.

We have a family of quantum GIT :

$$\widetilde{\mathscr{X}} := [\mathscr{S}(D) \times \Omega(d,n,D)/\mathbb{C}^{n-d}] \to \Omega(d,n,D)$$

It induces a projection $\mathscr{X} = \widetilde{\mathscr{X}}/\mathrm{Aut}(D) \to \mathscr{M}(d,n,D)$ □

Table of Contents

Embedding

The morphism

$$\Omega(d,n,D) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^d)^{epi} \xrightarrow{\ker(-)} \mathrm{Gr}(n-d,\mathbb{R}^n)$$

is an open immersion.

Embedding

The morphism

$$\Omega(d,n,D) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^d)^{epi} \xrightarrow{\ker(-)} \mathrm{Gr}(n-d,\mathbb{R}^n)$$

is an open immersion. Two advantages :

- $\mathrm{Gr}(n-d,\mathbb{R}^n)$ is a compact manifold ;
- The action of $\mathrm{Aut}(D)$ on the image of $\Omega(d,n,D)$ is easier to describe

Compactification

Theorem (B. ; 2021)

There exists a natural compactification $\overline{\mathscr{M}}$ *of* $\mathscr{M} = \mathscr{M}(d, n, D)$ *i.e. there exists a family* $\overline{\mathscr{X}} \to \overline{\mathscr{M}}$ *such that :*

1. *We have the following commutative diagram:*

$$\begin{array}{ccccc} & & \overline{\mathscr{X}} & \hookleftarrow & \mathscr{X} \\ & & \downarrow & & \downarrow \\ [\mathrm{Gr}(n-d, \mathbb{R}^n)/\mathrm{Aut}(D)] & \hookleftarrow & \overline{\mathscr{M}} & \hookleftarrow & \mathscr{M} \end{array}$$

2. *Over a point of* $\overline{\mathscr{M}} \setminus \mathscr{M}$*, we get a quantum toric stack with a degenerated combinatorial type of D (i.e. a subposet of D with the same 1-cones, stable by intersection and taking faces)*

Example of the moduli space of projective planes

$$\overline{\mathscr{M}(2,3,D_{\mathbb{P}^2})} = [\mathrm{Conv}([1,0,0],[0,1,0],[0,0,1])/\mathfrak{S}_3] \subset [\mathbb{RP}^2/\mathfrak{S}_3]$$

On each edges, we get a quotient of $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$ and on each vertices, we get a quotient of $\mathbb{C}^2 \times \mathbb{C}^*$.

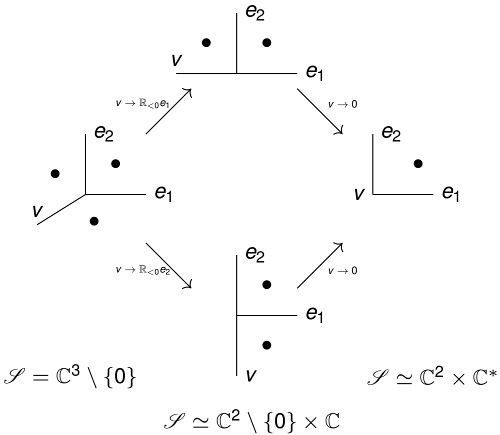


Table of Contents

Secondary fan

Question

If we fix the calibration $h : \mathbb{R}^n \to \mathbb{R}^d$, which combinatoric data lead to a quantum fan (Δ, h) ?

Secondary fan

Question

If we fix the calibration $h : \mathbb{R}^n \to \mathbb{R}^d$, which combinatoric data lead to a quantum fan (Δ, h) ?

Answer

Adaptation of the secondary fan in the quantum case.

It is a fan in $\mathbb{R}^{n-d}$ described by I.GELFAND, M.KAPRANOV, A.ZELEVINSKY which parameterizes GIT quotient for the action defined by (the Gale transform of) h on $\mathbb{C}^n$.

Example

Let $h : \mathbb{R}^4 \to \mathbb{R}^2$, be a calibration such that $h(e_3) = (\alpha, \beta), h(e_4) = (\gamma, \delta) \in \mathbb{R}^2_{<0}$.

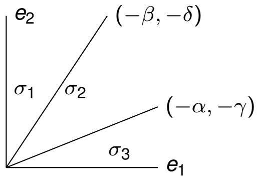


Figure: Secondary fan of h

	Classical	Quantum
σ_1, σ_3	(weighted) projective plane	quantum projective plane with one virtual generator
σ_2	Hirzebruch surface	quantum Hirzebruch surface

Differences rational/irrational

Rational case :
a point of the secondary fan $\rightsquigarrow$ a character of the (reductive) group G_h $\rightsquigarrow$ toric variety $(\mathbb{C}^n)^{ss} /\!\!/ G_h$
Quantum case :
Quotient by $\mathbb{C}^{n-d}$ (not reductive) $\rightsquigarrow$ the semi-stable and the stable loci are not what we can expect.

Differences rational/irrational

Rational case :
a point of the secondary fan $\rightsquigarrow$ a character of the (reductive) group G_h $\rightsquigarrow$ toric variety $(\mathbb{C}^n)^{ss} /\!/ G_h$
Quantum case :
Quotient by $\mathbb{C}^{n-d}$ (not reductive) $\rightsquigarrow$ the semi-stable and the stable loci are not what we can expect.

Example

Consider the action of $\mathbb{C}$ on $\mathbb{C}^2$ given by : $t \cdot (z_1, z_2) = (E(t)z_1, E(\alpha t)z_2)$ where $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$.
Let χ be the character $t \in \mathbb{C} \mapsto E(at) \in \mathbb{C}^*$. The G-semi-stable locus is :

$$(\mathbb{C}^2)^{ss}_\chi = \begin{cases} \emptyset & \text{if } a \notin \mathbb{N} + \mathbb{N}\alpha \\ (\mathbb{C}^*)^2 & \text{if } a \in \mathbb{N}_{>0} + \mathbb{N}_{>0}\alpha \\ \mathbb{C} \times \mathbb{C}^* & \text{if } a \in \mathbb{N}_{>0} \\ \mathbb{C}^* \times \mathbb{C} & \text{if } a \in \mathbb{N}_{>0}\alpha \\ \mathbb{C}^2 & \text{if } a = 0 \end{cases}$$

Differences rational/irrational

Rational case :
a point of the secondary fan $\rightsquigarrow$ a character of the (reductive) group G_h $\rightsquigarrow$ toric variety $(\mathbb{C}^n)^{ss}/\!\!/G_h$
Quantum case :
Quotient by $\mathbb{C}^{n-d}$ (not reductive) $\rightsquigarrow$ the semi-stable and the stable loci are not what we can expect.

Example

Consider the action of $\mathbb{C}$ on $\mathbb{C}^2$ given by : $t \cdot (z_1, z_2) = (E(t)z_1, E(\alpha t)z_2)$ where $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$.
Let χ be the character $t \in \mathbb{C} \mapsto E(at) \in \mathbb{C}^*$. The G-semi-stable locus is :

$$(\mathbb{C}^2)^{ss}_\chi = \begin{cases} \emptyset & \text{if } a \notin \mathbb{N} + \mathbb{N}\alpha \\ (\mathbb{C}^*)^2 & \text{if } a \in \mathbb{N}_{>0} + \mathbb{N}_{>0}\alpha \\ \mathbb{C} \times \mathbb{C}^* & \text{if } a \in \mathbb{N}_{>0} \\ \mathbb{C}^* \times \mathbb{C} & \text{if } a \in \mathbb{N}_{>0}\alpha \\ \mathbb{C}^2 & \text{if } a = 0 \end{cases}$$

$\Rightarrow$ We need a fully combinatorial approach for adapting this construction.

Combinatorial description

Let $\chi \in \mathbb{R}^{n-d}$ and $b \in \mathbb{R}^n$ such that $k^\top b = \chi$.
Consider the polyhedron

$$P_\chi := k^\top \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, h(e_i) \rangle \geq -b_i\} \subset \mathbb{R}^{n-d}$$

Suppose χ chosen such that P_χ is of maximal dimension $n - d$.

Combinatorial description

Let $\chi \in \mathbb{R}^{n-d}$ and $b \in \mathbb{R}^n$ such that $k^\top b = \chi$.
Consider the polyhedron

$$P_\chi := k^\top \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, h(e_i) \rangle \geq -b_i\} \subset \mathbb{R}^{n-d}$$

Suppose χ chosen such that P_χ is of maximal dimension $n-d$.

Virtual facets of P_χ = sets $F_{i,\chi} := \{x \in P_\chi \mid \langle x, h(e_i) \rangle = -b_i\}$ which are not genuine facets of P_χ.

Combinatorial description

Let $\chi \in \mathbb{R}^{n-d}$ and $b \in \mathbb{R}^n$ such that $k^\top b = \chi$.
Consider the polyhedron

$$P_\chi := k^\top \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, h(e_i) \rangle \geq -b_i\} \subset \mathbb{R}^{n-d}$$

Suppose χ chosen such that P_χ is of maximal dimension $n - d$.

Virtual facets of P_χ = sets $F_{i,\chi} := \{x \in P_\chi \mid \langle x, h(e_i) \rangle = -b_i\}$ which are not genuine facets of P_χ.

Construction of normal fan of a polyhedron $\rightsquigarrow$ fan Δ_χ
Virtual facets of P_χ $\rightsquigarrow$ virtual generators of Δ_χ

Combinatorial description

Let $\chi \in \mathbb{R}^{n-d}$ and $b \in \mathbb{R}^n$ such that $k^\top b = \chi$.
Consider the polyhedron

$$P_\chi := k^\top \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, \langle x, h(e_i) \rangle \geq -b_i\} \subset \mathbb{R}^{n-d}$$

Suppose χ chosen such that P_χ is of maximal dimension $n - d$.

Virtual facets of P_χ = sets $F_{i,\chi} := \{x \in P_\chi \mid \langle x, h(e_i) \rangle = -b_i\}$ which are not genuine facets of P_χ.

Construction of normal fan of a polyhedron $\rightsquigarrow$ fan Δ_χ
Virtual facets of P_χ $\rightsquigarrow$ virtual generators of Δ_χ

Lemma

We have the equality

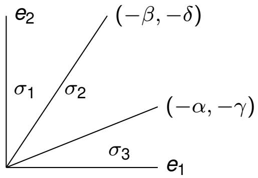
$$\{\chi \in \mathbb{R}^{n-d} \mid \dim(P_\chi) = n - d\} = \mathrm{Int}(\mathrm{Cone}(k^\top(e_1)), \dots, \mathrm{Cone}(k^\top(e_n))) =: \mathrm{Int}\,\mathrm{Cone}(k^\top).$$

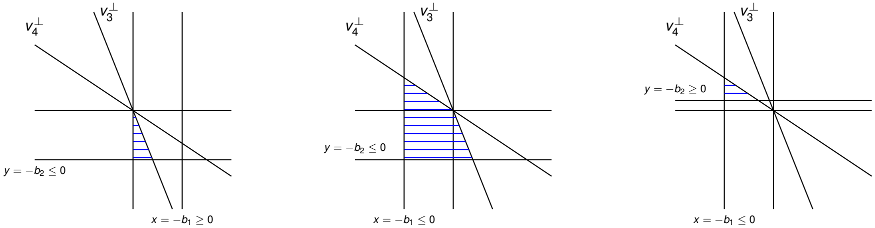
This cone is the support of a fan, called *secondary fan*.
The maximal cones of this fan are called "*chambers*" (the cone of the secondary fan corresponding to the fan $(\Delta, h, \mathcal{J})$ is denoted $\Gamma_{\Delta,\mathcal{J}}$) and their intersection are called "*walls*"

Example

Let $h : \mathbb{R}^4 \to \mathbb{R}^2$, be a calibration such that $h(e_3) = (\alpha, \beta), h(e_4) = (\gamma, \delta) \in \mathbb{R}^2_{<0}$. Then $k^\top : (x, y, z, t) \in \mathbb{R}^4 \mapsto (-\alpha x - \beta y + z, -\gamma x - \delta y + t) \in \mathbb{R}^2$ and hence

$$\mathrm{Cone}(k^\top) = \mathbb{R}^2_{\geq 0}.$$





Wall-crossing

The chamber $\Gamma_{\Delta,\mathcal{J}}$ can be written in the form $\Gamma_{\Delta,\emptyset} \times (\mathbb{R}_{\geq 0})^{\mathcal{J}}$. Let τ be a wall of $\Gamma_{\Delta,\mathcal{J}}$.

Wall-crossing

The chamber $\Gamma_{\Delta,\mathcal{J}}$ can be written in the form $\Gamma_{\Delta,\emptyset} \times (\mathbb{R}_{\geq 0})^{\mathcal{J}}$. Let τ be a wall of $\Gamma_{\Delta,\mathcal{J}}$. Two possibilities :

- $\tau = \Gamma_{\Delta,\emptyset} \times (\mathbb{R}_{\geq 0})^{\mathcal{J}\setminus\{i\}}$ [Flipping wall].
  τ is a wall between $\Gamma_{\Delta,\mathcal{J}}$ and $\Gamma_{\Delta',\mathcal{J}\setminus\{i\}}$ where Δ' is a star subdivision of Δ along $\mathrm{Cone}(h(e_i)) \cap \Gamma$.
  Exceptional locus of the birational morphism $\mathscr{X}_{\Delta',h,\mathcal{J}\setminus\{i\}} \dashrightarrow \mathscr{X}_{\Delta,h,\mathcal{J}}$ is of codimension 1.

Wall-crossing

The chamber $\Gamma_{\Delta,\mathcal{J}}$ can be written in the form $\Gamma_{\Delta,\emptyset} \times (\mathbb{R}_{\geq 0})^{\mathcal{J}}$. Let τ be a wall of $\Gamma_{\Delta,\mathcal{J}}$. Two possibilities :

- $\tau = \Gamma_{\Delta,\emptyset} \times (\mathbb{R}_{\geq 0})^{\mathcal{J}\setminus\{i\}}$ [Flipping wall].
  τ is a wall between $\Gamma_{\Delta,\mathcal{J}}$ and $\Gamma_{\Delta',\mathcal{J}\setminus\{i\}}$ where Δ' is a star subdivision of Δ along $\mathrm{Cone}(h(e_i)) \cap \Gamma$.
  Exceptional locus of the birational morphism $\mathscr{X}_{\Delta',h,\mathcal{J}\setminus\{i\}} \dashrightarrow \mathscr{X}_{\Delta,h,\mathcal{J}}$ is of codimension 1.

- $\tau = F \times (\mathbb{R}_{\geq 0})^{\mathcal{J}}$ [Divisorial wall].
  $\tau = \Gamma_{\Delta_0,h,\mathcal{J}}$ is a wall between two chambers $\Gamma_{\Delta,h,\mathcal{J}}$ and $\Gamma_{\Delta',h,\mathcal{J}}$
  - Δ_0 is not simplicial ;
  - $\Delta(1) = \Delta_0(1) = \Delta'(1)$;
  - Δ_0 is the coarsest common refinement of Δ and Δ' ;

$$\begin{array}{ccc} \mathscr{X}_{\Delta,h,\mathcal{J}} & \dashrightarrow & \mathscr{X}_{\Delta',h,\mathcal{J}} \\ & \searrow \;\; \swarrow & \\ & \mathscr{X}_{\Delta_0,h,\mathcal{J}} & \end{array}$$

Illustration of wall-crossing

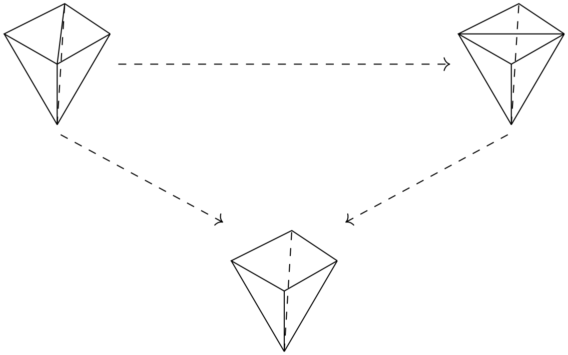


Figure: Divisorial wall

Table of Contents

Augmented moduli spaces

- parameterize equivalence classes of pairs (χ, h) where χ is in the secondary fan of $h : \mathbb{R}^n \to \mathbb{R}^d$ (where $h(e_i) = e_i$ for $1 \leq i \leq d$)

Definition

The augmented moduli space of toric stacks is the stack $\mathscr{A}(d, n)$ given by the stackification of the groupoid described by the equivalence relation $\sim$ on $U^{\mathrm{adm}} := \coprod_h \mathrm{Cone}(k^\top) \subset \mathbb{R}^{n-d} \times \mathbb{R}^{d(n-d)}$ defined by : $(\chi, h) \sim (\chi', h')$ if $\chi = \chi'$ and if there exists a quantum fan isomorphism $(\Delta_\chi, h) \overset{\simeq}{\rightrightarrows} (\Delta_{\chi'}, h')$.

Augmented moduli spaces

- parameterize equivalence classes of pairs (χ, h) where χ is in the secondary fan of $h : \mathbb{R}^n \to \mathbb{R}^d$ (where $h(e_i) = e_i$ for $1 \leq i \leq d$)

Definition

The augmented moduli space of toric stacks is the stack $\mathscr{A}(d, n)$ given by the stackification of the groupoid described by the equivalence relation $\sim$ on $U^{\mathrm{adm}} := \coprod_h \mathrm{Cone}(k^\top) \subset \mathbb{R}^{n-d} \times \mathbb{R}^{d(n-d)}$ defined by : $(\chi, h) \sim (\chi', h')$ if $\chi = \chi'$ and if there exists a quantum fan isomorphism $(\Delta_\chi, h) \overset{\simeq}{\rightrightarrows} (\Delta_{\chi'}, h')$.

Proposition

- *Let D be the combinatorial type of a quantum fan of $\mathbb{R}^d$ with n generators. Then there exists a substack $\mathscr{A}(D)$ of $\mathscr{A}(d, n)$ and a $\mathbb{R}^{n-d}$-fibration $\mathscr{A}(D) \to \mathscr{M}(d, n, D)$.*
- *If D is simplicial, $\mathscr{A}(D)$ is an open substack of $\mathscr{A}(d, n)$.*

Hence, $\mathscr{A}(d, n)$ contains a "thickening" of each moduli spaces $\mathscr{M}(d, n, D)$.

Connectedness of the augmented moduli space

Theorem (B., 2022)

Every "simplicial" point in the augmented moduli space can be linked by a continuous path to a quantum projective space.

Connectedness of the augmented moduli space

Theorem (B., 2022)

Every "simplicial" point in the augmented moduli space can be linked by a continuous path to a quantum projective space.

Sketch of proof.

Let $(\chi, h) \in U^{\mathrm{adm}}$ corresponding to the quantum fan (Δ, h). There are two steps :

- In the space $\Omega(d, n, \mathrm{comb}(\Delta))$, there exists an (Euclidean) open subset of calibration h such that (χ, h) can be linked with a projective space (through several wall-crossings).
- We conclude thanks to the (path-)connectedness of $\Omega(d, n, \mathrm{comb}(\Delta))$

□

Universal family of toric stacks over the augmented moduli space

Proposition

There exists a stack (over $\mathfrak{Man}_{\mathbb{R}}$) $\mathscr{E}(d,n)$ and a morphism $\mathfrak{p}\colon \mathscr{E}(d,n) \to \mathscr{A}(d,n)$ such that :

- *The fibers of $\mathfrak{p}$ are quantum toric stacks (given by a fan of $\mathbb{R}^d$ with n generators) i.e. is a universal family over $\mathscr{A}(d,n)$;*
- *There exist a (open if D is simplicial) substack $\mathscr{E}(D)$ of $\mathscr{E}(d,n)$ with a $\mathbb{R}^{n-d}$-fibration $\mathscr{E}(D) \to \mathscr{X}(d,n,D)$.*

In a diagram,

$$\begin{array}{ccccc}
\mathscr{X}(d,n,D) & \xleftarrow{\mathbb{R}^{n-d}\text{-fibration}} & \mathscr{E}(D) & \hookrightarrow & \mathscr{E}(d,n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{M}(d,n,D) & \xleftarrow[\mathbb{R}^{n-d}\text{-fibration}]{} & \mathscr{A}(D) & \hookrightarrow & \mathscr{A}(d,n)
\end{array}$$

where the fibers of the vertical arrows are quantum toric stacks.

Compactification of the augmented moduli space

Several steps :

- Schrink the secondary fans to their intersection with the unit ball $\overline{B(0,1)} \subset \mathbb{R}^{n-d}$ and hence considering the intersection (which is closed by taking the equivalence relation $\sim$).

$$U^{\mathrm{adm}} \cap \overline{B(0,1)} \times \mathbb{R}^{d(n-d)} \subset \overline{B(0,1)} \times \mathbb{R}^{d(n-d)} \subset \overline{B(0,1)} \times \mathrm{Gr}(n-d, \mathbb{R}^n).$$

- Take the invariant closure K of this space in $\overline{B(0,1)} \times \mathrm{Gr}(n-d, \mathbb{R}^n)$.
- Build the stack $\mathscr{K}(d,n)$ by stackification of the groupoid given by the equivalence relation $\sim$ on K.
- Idem for the universal family.

To sum up,

Theorem (B., 2022, "Big Toric moduli conjecture")

Let $n \geq d$ two integers. There exists a compact stack (over $\mathfrak{Man}_{\mathbb{R}}$) $\mathscr{K}(d,n)$ and a stack morphism $\overline{\mathscr{E}(d,n)} \to \mathscr{K}(d,n)$ such that :

- *For every combinatorial type D of a complete fan of $\mathbb{R}^d$ with n generators,there exists a closed substack $\mathscr{K}(D)$ of $\mathscr{K}(d,n)$ and a $\mathbb{R}^{n-d}$-fibration $\mathscr{K}(D) \to \overline{\mathscr{M}(d,n,D)}$;*
- *This fibration extends to a fibration between the family over $\mathscr{K}(D)$ and the family over $\overline{\mathscr{M}(d,n,D)}$.*

Table of Contents

Perspectives

- Topology and Geometry of the spaces $\mathscr{M}(d, n, D)$ and $\overline{\mathscr{M}(d, n, D)}$:
  Regular/Singular locus, homotopy groups, cohomology group, ...

- Irrational combinatorial type:
  Study the combinatorial type which can be realized only over an extension of $\mathbb{Q}$ (example for polytopes are given by Perles) and their links with (augmented) moduli spaces.

- Quantum toric stacks as limits of (DM) toric stacks and application to mirror symmetry.

- Homological Mirror Symmetry for quantum toric stacks

Thank you for your attention !